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Solution of the Elliptic Rendezvous Problem with the Time as Independent Variable

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An explicit solution is given for the linearized motion of a chaser in a close neighborhood of a target in an elliptic orbit. The solution is a direct generalization of the Clohessy–Wiltshire equations that are widely used for circular orbits. In other words, when the eccentricity is set equal to zero in the new formulas, the well-known Clohessy–Wiltshire formulas are obtained. The solution is completely explicit in the time. As a starting point, a closed-form solution is found of the de Vries equations of 1963. These are the linearized equations of elliptic motion in a rotating coordinate system, rotating with a variable angular velocity. This solution is shown to be obtained simply by taking the partial derivatives, with respect to the orbit elements, of the two-body solution in polar coordinates. When four classical elements are used, four linearly independent solutions of the de Vries equations are obtained. However, the classical orbit elements turn out to be singular for circular orbits. The singularity is removed by taking appropriate linear combinations of the four solutions. This gives a 4 by 4 fundamental solution matrix *R* that is nonsingular and reduces to the Clohessy–Wiltshire solution matrix when the eccentricity is set equal to zero.

Introduction

THE ability to describe the motion of one space vehicle with respect to another space vehicle is of great practical importance to the spacecraft mission analyst. Immediate applications of such a description of the motion include the spacecraft rendezvous problem and the spacecraft stationkeeping problem. Both of these problems really fall into a more general class of trajectory problems commonly known as relative motion problems. By rendezvous, it is meant that two space vehicles are brought to within a close proximity of each other with low relative velocity. This is accomplished by one vehicle (the chaser) performing a maneuver (or sequence of maneuvers) until some desired position relative to the other vehicle (the target) is achieved.

The purpose of this paper is to examine the relative motion problem with a particular emphasis on targeting in elliptical orbits. A new solution to the linearized, elliptical equations of relative motion is developed. The solution to these equations is developed in terms of the partial derivatives of the polar coordinates with respect to the orbital elements. For state propagation, this development includes the derivation of the state transition matrix in a rotating, relative motion reference frame. With use of the results obtained from the state transition matrix, the targeting equations for the rotating, relative motion reference frame are solved.

Historically, the equations of motion are close to some equations in the theory of the moon developed by Hill[1] in the 19th century. Therefore, the equations of motion governing this problem are sometimes referred to as Hill's equations. However, as far as manufactured Earth satellites are concerned, it seems to be Lawden[2] (1954) who first formulated the equations of relative motion using normalized coordinates and true anomaly as the independent variable. As far as it can be determined, Lawden was the first to give the complete analytical solution, valid for elliptical orbits.[3] He again used true anomaly as the independent variable, but his solution is not the most simple, mainly because of the presence of some definite integrals.

Similar solutions, valid for elliptical orbits, have also been obtained by Hempel and Tschauner,[4] Tschauner and Hempel,[5,6] Tschauner,[7] Hempel,[8] and Tschauner[9] between 1964 and 1967. Their solution was later used and extended by several authors, such as Carter and Humi,[10] Carter,[11] Wolfsberger et al.,[12] and Kelly.[13] Note that all of these authors always used true anomaly as the independent variable.

In the present paper, a new general solution to the equations describing the motion of a chaser spacecraft with respect to a target spacecraft in an elliptical orbit is developed. This solution uses the time as the independent variable, and it is presented in a concise general form using the polar coordinates of the target space vehicle. It is believed that this solution has not been previously given in the literature.

The motivation of our work can be found in that all of the previous works use the true anomaly as the basic variable, or use a change of scale of the dependent variables, or contain some integrals that may be difficult to handle. Our goal was to find a formulation that is not affected by these three problems and that reduces to the Clohessy–Wiltshire[14] equations when the eccentricity is set equal to zero.

Some History of Elliptic Rendezvous

The two names that are by far the best known in the theory of rendezvous and relative motion are Clohessy and Wiltshire.[14] Their basic equations are widely used in industry and in NASA centers. This is mostly due to the simplicity of the theory because they only consider the case of a circular target orbit. In this case, we obtain a system of linear equations with constant coefficients, which can be solved by elementary methods and is well understood.

On the other hand, the corresponding theory for elliptic target orbits is much less known and in fact rarely used in practice. This is due to the additional complexities of this theory because the linearization results in a system of equations with time-varying coefficients. This is, however, a paradoxical situation. The solution for the elliptic relative equation of motion is available in the literature, even before the start of the space era, about 6 years before the Clohessy–Wiltshire paper, in a paper by Lawden[2] written in 1954.

The Lawden[2] paper definitely contains the essence of the solution of the elliptic relative motion. However it may not appear as attractive because of the double change of variables that is used: The coordinates are normalized by dividing them by the radius-vector of the target, and the true anomaly of the target orbit is used as an independent variable. Note that Lawden's solution was later reproduced in his book on optimal trajectories.[3]

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This change of variables has in fact a long history in celestial mechanics. The coordinate normalization was already found, about a century earlier. The complete transformation (dimensionless coordinates and switch to the true anomaly) is well known in the theory of the elliptic restricted three-body problem. It was introduced by Nechville.[15]

A number of authors have since published solutions of the elliptic relative motion, all based on the use of the true anomaly, similar to Lawden's[2] solution, although very few of them seemed to be aware of Lawden's work.

On the other hand, in 1963, de Vries[16] published the exact linearized equations of the elliptic rendezvous, using the ordinary time as independent variable. He gave approximate solutions but not the complete analytical solution. This will be the subject of our present paper. Many other papers have given approximate solutions, especially expansions in the eccentricity.

Along the same line of thought as Lawden, we find in the mid-1960s a series of some five articles in German, by Hempel and Tschauner,[4] Tschauner and Hempel,[5,6] Tschauner,[7] and Hempel.[8] A short article also appeared in English[9] in 1967, but this may have been too concise to be noticed by the general American astrodynamics community. All of these publications use a change of independent variable, generally to the true anomaly, although at least one of their papers is based on the use of the eccentric anomaly (Tschauner and Hempel[6]).

References 4–9 have been at the origin of several other papers on the subject in Germany. In particular, the paper by Wolfsberger et al.[12] contains an appendix with the summary of formulas required for a rendezvous with an elliptic target orbit. Again, these formulas assume that the true anomaly is used as an independent variable. Two other papers that both use the true anomaly as independent variable have recently been published by Garrison et al.[17] in 1995 and Carter[18] in 1998. In 2000, Melton[19] published a solution for the elliptic relative motion, which is completely explicit in the time but in the form of series expansion in the eccentricity. Finally, even more recently, Wiesel[20] published a new approach to relative motion, which allows for the inclusion of perturbations.

Fundamental Equations of Motion for Elliptic Rendezvous

The dynamic system of interest consists of two space vehicles orbiting about a common central body, the target space vehicle (subscript 1) and the chaser space vehicle. The two space vehicles are on nearby unperturbed elliptical orbits. The motion of the chaser is restricted to the plane of the orbit of the target. The out-of-plane motion will be considered in Appendix A.

The rotating coordinate system always has its origin at the target spacecraft. The x axis of this system is along the radius vector, which connects the target to the central body, positive in the direction pointing away from the central body. The y axis of this system is perpendicular to the x axis, positive in the direction of the motion. This system is obtained from the inertial system by a rotation through the angle θ and a translation of distance r_1. This coordinate system is commonly called a local vertical–local horizontal (LVLH) reference frame (Fig. 1). The frame x, y, centered at the target, is the basic rotating reference frame. Quantities without subscripts will generally be referring to this coordinate system. The coordinate transformation between the two frames is defined by

$$x_{12} = (x + r_1)\cos\theta - y\sin\theta \tag{1}$$

$$y_{12} = (x + r_1)\sin\theta - y\cos\theta \tag{2}$$

where $\theta = \omega + \nu$.

The velocity components satisfy the relation

$$\dot{x}_{12}^2 + \dot{y}_{12}^2 = [(\dot{x} + \dot{r}_1) - y\dot{\theta}]^2 + [\dot{y} + (x + r_1)\dot{\theta}]^2 \tag{3}$$

Considering that $r_2^2 = (r_1 + x)^2 + y^2$ and using the Taylor series

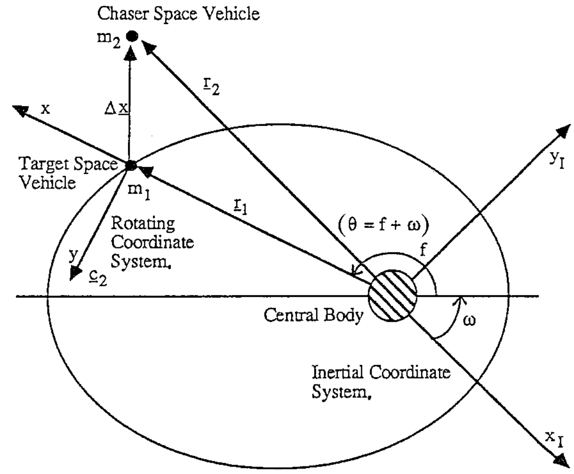

Fig. 1 Inertial reference frame (x_1, y_1) centered at the center of the Earth; variables r_1 and r_2 are radius vectors of the target and the chaser, respectively.

expansion

$$1/r_2 = 1/r_1\left\{1 - (x/r_1) + \left[(2x^2 - y^2)\big/2r_1^2\right]\right\} \tag{4}$$

we find that the Lagrangian for the motion of the chaser is

$$L_2 = \tfrac{1}{2}[(\dot{x} - y\dot{\theta})^2 + (\dot{y} + x\dot{\theta})^2] + \left(\mu\big/r_1^3\right)[x^2 - (y^2/2)] \tag{5}$$

The corresponding Euler–Lagrange equations are

$$\ddot{x} - 2\dot{\theta}\dot{y} - \dot{\theta}^2 x - \ddot{\theta}y = +2\left(\mu\big/r_1^3\right)x \tag{6}$$

$$\ddot{y} + 2\dot{\theta}\dot{x} - \dot{\theta}^2 y + \ddot{\theta}x = -\left(\mu\big/r_1^3\right)y \tag{7}$$

These are the linearized equations for the motion of the chaser relative to a target in an elliptical orbit. The fundamental equations of motion have already been introduced by many authors, for instance, De Vries.[16]

In the preceding system of differential equations, the different coefficients have to be considered as known functions of time. In particular,

$$r_1 = a(1 - e^2)/(1 + e\cos v) \tag{8}$$

$$\dot{\theta} = \dot{f} = C\big/r_1^2 \tag{9}$$

$$\ddot{\theta} = -2\mu e\sin v\big/r_1^3 \tag{10}$$

where C is the angular momentum $(\mu p)^{\frac{1}{2}}$.

Analytical Solution of the Equations of Motion

It is theorized that because a complete analytical solution to the nonlinear system of differential equations is known, a complete analytical solution to the linearized system of differential equations (6) and (7) can also be found. In this section, we will show a general form for this analytical solution. Some additional details are given in Appendix B.

Let $\boldsymbol{r}_1 = \boldsymbol{r}_1(\alpha, t)$ be the motion of the target relative to the inertial frame of reference. Here α stands for the set of four classical orbit elements, a, e, M_0, and ω. We will assume that the chaser has a similar Keplerian orbit, except that one of the four elements α has to be changed by a small amount $\Delta\alpha$. However, first, we represent the motion of the target (in the inertial frame) by its polar coordinates,

$$x_1 = r_1(\alpha)\cos\theta(\alpha), \qquad y_1 = r_1(\alpha)\cos\theta(\alpha) \tag{11}$$

where the notations $r_1(\alpha)$ and $\theta(\alpha)$ remind us that the polar coordinates r_1 and θ are functions of any one orbit element α. The motion of the chaser can be considered as a similar Keplerian motion, but with a slightly changed value ($\alpha + \Delta\alpha$) of the orbit element α. For the solution of the relative equations of motion, a variation along the radial direction gives x, whereas a variation in the transverse direction gives y. Therefore, four linearly independent solutions can be written as

$$x = \frac{\partial r_1}{\partial \alpha} \cdot \Delta\alpha, \qquad \dot{x} = \frac{\partial \dot{r}_1}{\partial \alpha} \cdot \Delta\alpha$$

$$y = r_1(\alpha)\frac{\partial \theta}{\partial \alpha} \cdot \Delta\alpha, \qquad \dot{y} = \left(\dot{r}_1 \frac{\partial \theta}{\partial \alpha} + r_1 \frac{\partial \dot{\theta}}{\partial \alpha}\right)\Delta\alpha \tag{12}$$

where $\Delta\alpha$ an arbitrary small constant. Thus, all we need is the explicit expressions for the partial derivatives of the polar coordinates r, and θ with respect to the classical orbit elements. These expressions can be directly obtained from classical textbooks. However, two of the four solutions can be obtained by some simple reasoning.

The simplest case deals with the orbit element ω, the argument of the perigee. We know that the radius vector r depends only on a, e, and M_0 but not ω. On the other hand, we have $\theta = \omega + v$, where the true anomaly v also depends only on the same three elements, α, e, M_0. Therefore, $\partial r_1/\partial\omega = 0$ and $\partial\theta/\partial\omega = 1$. This gives us the simplest of the four independent solutions of the linear equations of relative motion:

$$x = 0, \qquad y = r_1, \qquad \dot{x} = 0, \qquad \dot{y} = \dot{r}_1 = \sqrt{(\mu/p)}e \sin v \tag{13}$$

It turns out that a second solution can also be obtained very easily. This is the solution related to the variation in M_0, the mean anomaly at epoch. Because of the simple relation between M and M_0, it is sufficient to take partial derivatives with respect to M rather than M_0. However, in the two-body problem, this is equivalent to taking time derivatives, except for the mean motion factor n (because $M \approx nt$). Therefore, the corresponding particular solution for the relative notion is

$$x = \frac{\partial r_1}{\partial M} = \frac{\dot{r}_1}{n}, \qquad y = r_1 \frac{\partial \theta}{\partial M} = \frac{r_1 \dot{\theta}}{n} \tag{14}$$

As for the two remaining solutions related to the variations in a and e, they require a little more work. However, these partials can be derived from the known two-body formulas; some of the details are summarized in Appendix C.

The four independent solutions for the relative motion equations, as well as the time derivative of these solutions (the matrix R), are

$$R = \left[\begin{array}{c|c|c|c}
\dfrac{r}{a} - \dfrac{3n(t-t_0)e\sin v}{2\sqrt{1-e^2}} & -\cos v & \dfrac{e\sin v}{\sqrt{1-e^2}} & 0 \\
-\dfrac{3}{2}n(t-t_0)\sqrt{1-e^2}\cdot\left(\dfrac{a}{r}\right) & \left(1+\dfrac{r}{p}\right)\sin v & \sqrt{1-e^2}\cdot\left(\dfrac{a}{r}\right) & \dfrac{r}{a} \\
\dfrac{-ne\sin v}{2\sqrt{1-e^2}} - \dfrac{3}{2}(t-t_0)e\cos v\cdot\left(\dfrac{na}{r}\right)^2 & n\sin v\sqrt{1-e^2}\cdot\left(\dfrac{a}{r}\right)^2 & en\cos v\cdot\left(\dfrac{a}{r}\right)^2 & 0 \\
\dfrac{3}{2}(t-t_0)e\sin v\left(\dfrac{na}{r}\right)^2 - \dfrac{3}{2}\sqrt{1-e^2}\cdot\left(\dfrac{na}{r}\right) & n\sqrt{1-e^2}\left(1+\dfrac{r}{p}\right)\left(\dfrac{a}{r}\right)^2\cos v + \dfrac{en\sin^2 v}{\sqrt{(1-e)^3}} & -en\sin v\cdot\left(\dfrac{a}{r}\right)^2 & \dfrac{en\sin v}{\sqrt{1-e^2}}
\end{array}\right] \tag{15}$$

Each column of the matrix corresponds to one of the orbit elements. However, to increase the numerical stability of the calculations with these solutions, the second to forth columns have all been divided by the semimajor axis, so that dimensionless quantities have been obtained.

Note that the matrix $[R]$ is a fundamental matrix of solutions to the dynamic system described by Eqs. (6) and (7). Specifically, each column of $[R]$ is a linearly independent solution of these equations.

Connection with the Circular Clohessy–Wiltshire[14] Solution

In this section we point out that the Clohessy–Wiltshire[14] equations of circular relative motion can be derived from Eqs. (6) and (7) by making some additional simplifying assumptions. Specifically, if it is assumed that the target spacecraft is in a circular orbit, the following relationships hold from Kepler's third law:

$$n^2 a^3 = \mu, \qquad a = r_1, \qquad \dot{\theta}^2 = \mu/r_1^3 \tag{16}$$

where the angular velocity θ is a constant. By substitution of Eqs. (16) into Eqs. (6) and (7) and simplification, the following equations of motion are obtained:

$$\ddot{x} - 2n\dot{y} - 3n^2 x = 0 \tag{17}$$

$$\ddot{y} + 2n\dot{x} = 0 \tag{18}$$

The are the famous Clohessy–Wiltshire equations, linearized about the circular orbit of the target spacecraft. Thus, the well-known Clohessy–Wiltshire equations are just a special case of our elliptic equations (6) and (7). We will not repeat the analytical solution here. However, this solution can be obtained by setting the eccentricity equal to zero in the R matrix [Eq. (15)]. We find

$$R_0 = \left[\begin{array}{c|c|c|c}
1 & -\cos M & 0 & 0 \\
-\frac{3}{2}n(t-t_0) & 2\sin M & 1 & 1 \\
0 & n\sin M & 0 & 0 \\
-\frac{3}{2}n & 2n\cos M & 0 & 0
\end{array}\right] \tag{19}$$

where $M = M_0 + n(t - t_0)$. Unfortunately, note that, when the eccentricity of the reference orbit approaches zero, the matrix $[R]$ becomes singular. In other words, as the reference trajectory approaches the circular orbit, the matrix $[R]$ admits only three linearly independent solutions to the equations of motion. This means that for the circular target reference orbit our solution fails. Consequently, in the limit $e = 0$, our new elliptic solution matrix R does not give the complete solution to the Clohessy–Wiltshire equations of motion. This problem will be addressed in the following section.

Solution of the Zero-Eccentricity Difficulty

As already said, when the eccentricity e goes to zero, the two last columns of the fundamental matrix R become identical, and therefore, the matrix becomes singular.

To remove this problem for the circular limiting case, we will replace the third column by a linear combination of the third and forth columns. The new solution will be

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\sqrt{1-e^2}}{aen}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \end{bmatrix} - \frac{1}{ep}\begin{bmatrix} 0 \\ r \end{bmatrix} \tag{20}$$

The new third column of the matrix will then be:

$$\begin{bmatrix} x = \sin v \\ y = \dfrac{r}{p}\cos v \cdot (2 + e\cos v) \\ \dot{x} = +\dfrac{\sqrt{\mu p}}{r^2}\cos v \\ \dot{y} = -\sqrt{\dfrac{\mu p}{r^2}} e \sin v \cdot \left(1 + \dfrac{r^2}{p^2}\right) \end{bmatrix} \tag{21}$$

This results in the new fundamental solution matrix. The other three columns are unchanged.

It is still valid and nonsingular in the case of exactly circular orbits. In fact, in this case it reduces to the well-known Clohessy–Wiltshire[14] matrix:

$$R_0 = \left[\begin{array}{c|c|c|c} +1 & -\cos M & \sin M & 0 \\ -3n(t-t_0)/2 & 2\sin M & 2\cos M & 1 \\ 0 & n\sin M & n\cos M & 0 \\ -3n/2 & 2n\cos M & -2n\sin M & 0 \end{array}\right] \tag{22}$$

The first, second, and fourth columns correspond to the partial derivatives with respect to a, e and ω respectively. M is the mean anomaly.

To summarize, we have now been able to obtain a fundamental matrix R of solutions to the linearized elliptic relative motion equations. This matrix is always non-singular, even in the limiting case of circular target orbits. In fact, its determinant is $-n^2/2$, where n the mean motion is. Our next important goal is to derive the state transition matrix, but this requires the inversion of the fundamental matrix R, which will be the object of the next section.

Inversion of the Matrix *R*

The matrix R can be inverted with the use of the standard rules, taking the 3 by 3 cofactors of all 16 terms and then dividing by the determinant. However, because of the amount of labor involved in these operations, it is worthwhile taking a closer look at the matrix R and its special properties with the hope of finding a few shortcuts and peculiarities that would make the matrix inversion simpler.

The first obvious property is that the matrix R contains the time t explicitly, only in a linear fashion, only in the first column. However there is an even deeper property there. Let us designate the first column of R by $M_1 - 3n(t - t_0)M_1'/2$, where M_1 is the column of terms not containing t and M_1' the column of terms that all four have the common factor $3n(t - t_0)/2$:

$$M_1' = [(a/b)e\sin v;\ (b/r);\ n(a/r)^2 e\cos v;\ -n(a/r)^2 e \sin v]^T \tag{23}$$

where the T indicates a transpose. We also designate the three other columns of R by M_2, M_3, and M_4.

We can now verify that M_1' is a linear combination of the two last columns

$$M_1' = e(1-e^2)^{\frac{1}{2}} M_3 + (1-e^2)^{\frac{3}{2}} M_4 \tag{24}$$

This has important consequences. In particular, we know that a determinant is not changed if we add or subtract any linear combination of columns to another column. Consequently, the determinant of R is the same as the determinant of the matrix $R' = (M_1;\ M_2;\ M_3;\ M_4)$. We actually find that this determinant is constant and equals $-n^2/2$.

The same property can also be used to simplify the computation of the cofactors. Property (24) also has a consequence the fact that the inverse R^{-1} is linear in time. (In other words, no t^2 or t^3 terms are present.) In fact, the inverse R^{-1} has only two columns of terms containing t explicitly: the third and fourth columns.

In more general notations, the matrix R is of the form $A + Bt$, and its inverse R^{-1} of the form $X + Yt$, where A, B, X, and Y are all 4 by 4 matrices. In fact, here $A = R' = (M_1;\ M_2;\ M_3;\ M_4)$. We propose to invert A first. Then the inverse $R^{-1} = X + Yt$ is given by

$$X = A^{-1}, \qquad Y = -A^{-1}BA^{-1} \tag{25}$$

Incidentally, notice that our matrix $R = A + Bt$ has the property that $BA^{-1}BA^{-1} = 0$.

For the construction of the inverse matrix A^{-1}, it could, for instance, be done easily in the form of block matrices, each block being of dimension 2 by 2. We see, indeed, that A can be rearranged in such a way that the upper left quarter is

$$A' = \begin{bmatrix} \dfrac{r}{a} & 0 \\ 0 & \dfrac{r}{a} \end{bmatrix} \tag{26}$$

This is, thus, diagonal and has a trivial inverse. This greatly simplifies the 2 by 2 block approach to the inversion of A. The final result that is obtained for R^{-1} is still too long to be written out in one piece. Thus, we write the inverse of R in terms of three smaller 4 by 2 matrices I, J, and K:

$$R^{-1} = [I;\ J + 3(t - t_0)K]^T \tag{27}$$

$$I = \left[\begin{array}{c|c} 2\left(\dfrac{a}{r}\right)^2\left(1 + \dfrac{p}{r}\right) & \left(\dfrac{a}{r}\right)[3\cos v + e(2 + \cos^2 v)] \\ -2\left(\dfrac{a}{r}\right)^2 e\sin v & -e\sin v\dfrac{a}{p}[e + \cos v(2 + e\cos v)] \\ \dfrac{2e\sin v}{n\sqrt{1-e^2}} & \dfrac{\sqrt{1-e^2}}{n}\sin v \\ 2\left(\dfrac{a}{r}\right)\dfrac{\sqrt{1-e^2}}{n} & \dfrac{\sqrt{1-e^2}}{n}\left[\cos v + \dfrac{r}{p}(e + \cos v)\right] \end{array}\right] \tag{28}$$

$$J = \begin{bmatrix} \frac{-a}{p}\sin v[e^2 + (1+e\cos v)(3+e\cos v)] & \frac{-a^2}{pr} e\sin v\left(2+\frac{p}{r}+\frac{r}{p}\right) \\ \left(\frac{a}{p}\right) e\sin^2 v\cdot(2+e\cos v) & \left(\frac{a}{p}\right)^2 [1+e\cos v - e^2(e\cos^3 v + 2\cos^2 v - 1)] \\ \frac{r\sqrt{1-e^2}}{pn}[\cos v + e(\cos^2 v - 2)] & \frac{e\cos v - 1}{n\sqrt{1-e^2}}\cdot\left(1+\frac{r}{p}\right) \\ \frac{-\sqrt{1-e^2}}{n}\sin v\cdot\left(1+\frac{r}{p}\right) & \frac{-e\sin v}{n\sqrt{1-e^2}}\cdot\left(1+\frac{r}{p}\right) \end{bmatrix} \tag{29}$$

$$K = \begin{bmatrix} \frac{ne}{\sqrt{1-e^2}}\left(\frac{a}{r}\right)\left(1+\frac{p}{r}\right) & \frac{n}{\sqrt{(1-e^2)^3}}\left(\frac{a}{r}\right)^2\left(1+\frac{p}{r}\right) \\ \frac{-ne^2\sin v}{\sqrt{1-e^2}}\cdot\left(\frac{a}{r}\right)^2 & \frac{-ne\sin v}{\sqrt{(1-e^2)^3}}\cdot\left(\frac{a}{r}\right)^2 \\ \frac{e^2\sin v}{1-e^2} & \frac{e\sin v}{(1-e^2)^2} \\ \frac{ae}{r} & \frac{a}{(1-e^2)r} \end{bmatrix} \tag{30}$$

As expected, the preceding inverse matrix has no difficulties with circular orbits. In fact, when the eccentricity becomes zero, we obtain a much more simple matrix, which is the inverse of R as given in Eq. (19) for the Clohessy–Wiltshire case:

$$R_0^{-1} \begin{bmatrix} 4 & 0 & 0 & \frac{2}{n} \\ 3\cos M & 0 & \frac{\sin M}{n} & \frac{2\cos M}{n} \\ -3\sin M & 0 & \frac{\cos M}{n} & \frac{-2\sin M}{n} \\ 6n(t-t_0) & 1 & \frac{-2}{n} & 3(t-t_0) \end{bmatrix} \tag{31}$$

Equations for Elliptic Targeting

The principal goal in the developments of the preceding sections was to establish a relationship of the form

$$\boldsymbol{x}(t_2) = \Phi(t_2, t_1)\cdot\boldsymbol{x}(t_1) \tag{32}$$

between two relative state vectors of the chaser spacecraft, at times t_1 and t_2.

The expression $\Phi(t_2, t_1)$ is known as the state transition matrix, and it can be found by using the results of the preceding sections. Specifically, if a fundamental matrix $R(t)$ of solutions is known to some dynamic system, then

$$\Phi(t_2, t_1) = [R(t_2)][R(t_1)]^{-1} \tag{33}$$

We will now apply this result to the elliptic targeting problem, which can be defined as follows. Given some initial state $\boldsymbol{x}_1$ at t_1, we need to compute the maneuver at time t_1 that will cause the chaser to reach the position $\boldsymbol{x}_2$ at time t_2. Then we usually need to compute the maneuver at time t_2, which nulls the relative rates of the two spacecraft.

One of the most important given parameters of the problem is the time $(t_2 - t_1)$ between the two maneuvers. The principal unknowns are the postburn velocity $(\dot{x}_1, \dot{y}_1)$ at time t_1 and the arrival velocity $(\dot{x}_2, \dot{y}_2)$ at time t_2.

In regards to terminology and general practice, the maneuver at time t_1 is commonly called the intercept maneuver (t_1 maneuver). The maneuver at time t_2 is commonly called the nulling maneuver (t_2 maneuver) and nulls the relative rates of the chaser spacecraft with respect to the target spacecraft in the moving frame of reference.

To see the solution to this problem, we first rewrite Eq. (32) in component form:

$$x_2 = Ax_1 + By_1 + C\dot{x}_1 + D\dot{y}_1 \tag{34a}$$

$$y_2 = Ex_1 + Fy_1 + G\dot{x}_1 + H\dot{y}_1 \tag{34b}$$

$$\dot{x}_2 = Ix_1 + Jy_1 + K\dot{x}_1 + L\dot{y}_1 \tag{34c}$$

$$\dot{y}_2 = Mx_1 + Ny_1 + O\dot{x}_1 + P\dot{y}_1 \tag{34d}$$

Because the postmaneuver velocities $(\dot{x}_1, \dot{y}_1)$ are unknown, we solve Eqs. (34a) and (34b) simultaneously for the velocity components:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = (CH - DG)^{-1}\begin{bmatrix} H & -D \\ -G & C \end{bmatrix}\begin{bmatrix} Q \\ R \end{bmatrix} \tag{35}$$

where

$$Q = x_2 - Ax_1 - By_1, \qquad R = y_2 - Ex_1 - Fy_1 \tag{36}$$

Then the premaneuver velocities at t_2 can be found by substituting the solutions of Eqs. (35) and (36) into Eqs. (34c) and (34d), the postmaneuver velocities at t_2 being zero. The t_2 maneuver is, thus, a burn with delta V:

$$\Delta\boldsymbol{V} = (-\dot{x}_2, -\dot{y}_2) \tag{37}$$

The preceding equations may be considered as the elliptic generalization of the well-known Clohessy–Wiltshire[14] targeting to a circular orbit.

Note that the present targeting equations are only valid for small distances because of the linearization that is inherent in the derivation of the theory. In a more general sense, the targeting problem may be considered as a two-point boundary value problem. If the distances are large, it is customary to use the Lambert theorem. Lambert targeting does not suffer from the linear approximation problems, but it is much more computationally intensive.

Conclusions

We gave an explicit solution for the linearized motion of a chaser in a close neighborhood of a target in an elliptic orbit, which is a direct generalization of the Clohessy–Wiltshire formulas. The solution is in the form of a 4-by-4 fundamental matrix R which is non-singular and reduces to the Clohessy–Wiltshire solution matrix when the eccentricity of the target orbit goes to zero. Next, we constructed the explicit inverse of the matrix R, because we want the state transition matrix, which is the product of R and its inverse. The determinant of the R matrix is constant and always different from zero. In the last section of the paper, we then applied our new state transition matrix to develop the corresponding targeting equations, that is the formulas needed to find the velocity change required by the chaser to reach the target in a given time interval.

Appendix A: Linearized Out-of-Plane Motion

In the main body of our text, we have only studied the linearized motion in the plane of the target orbit. This is justified by that the motion in the third dimension (out of the plane of the target orbit) uncouples from the in-plane motion. We will show here that there is a simple analytic solution for the out-of-plane motion. The linearized equation of motion of the chaser in the third dimension is similar to Eq. (7), except that it is not affected by the rotation:

$$\ddot{z} = -(\mu/r^3)z \tag{A1}$$

where r is the radius vector of the target, a known periodic function of the time or true or eccentric anomaly $E : r = a(1 - e\cos E)$. This is the linear homogeneous Hill equation that was described by Encke in 1854, as well as Hill, in his famous memoir of 1874.

For the solution of Eq. (A1), it is known from the theory of the two-body problem that we have the following independent particular solutions:

$$z_1 = 1\big/\sqrt{n}(\cos E - e) = \sqrt{(an/\mu)}r\cos v \tag{A2}$$

$$z_2 = 1\big/\sqrt{n}(\sin E) = \sqrt{an/\mu(1 - e^2)}r\sin v \tag{A3}$$

where E and v are the eccentric and true anomaly, respectively. We selected the constant factor in front of z_1 and z_2 in such a way as to obtain a Wronskian determinant equal to 1:

$$\det\begin{bmatrix} z_1 & z_2 \\ \dot{z}_1 & \dot{z}_2 \end{bmatrix} = +1 \tag{A4}$$

It can be quickly verified by direct substitution in Eq. (A1) that we truly have two solutions. It is sufficient to know the time derivative of the eccentric anomaly E: $\dot{E} = na/r$.

Thus, the fundamental solution matrix (as well as the inverse) for the out-of-plane motion is given by

$$R_Z(t) = \begin{bmatrix} z_1 & z_2 \\ \dot{z}_1 & \dot{z}_2 \end{bmatrix}, \qquad R_2^{-1}(t) = \begin{bmatrix} z_2 & -z_2 \\ -\dot{z}_1 & z_1 \end{bmatrix} \tag{A5}$$

The state transition matrix Φ_z is the product of the two preceding matrices, evaluated at the different instants of time t_1 and t_2. The complete 6 by 6 state transition matrix Φ consists of two diagonal blocks, one a 4 by 4 and the other a 2 by 2 matrix,

$$\Phi(t_1, t_2) = \begin{bmatrix} \Phi_{xy} & 0 \\ 0 & \Phi_Z \end{bmatrix} \tag{A6}$$

The 4 by 4 matrix for the in-plane relative motion was given in Eqns. (22) and (31).

Appendix B: Some Details on the Explicit Solution of the de Vries[16] Equations

A few additional comments on the basic ideas, as well as the algebra involved in solving the equations of motion, are given. First, the equations of relative motion are the linearization of a nonlinear system, which is the explicitly integrable Kepler problem. Then, there is a general principle in dynamics that when the nonlinear system is integrable, we can write the explicit solution of the associated linearized system. The rule is rather simple: The solution of the linearized system is obtained by taking the partial derivatives of the solution of the nonlinear system with respect to the constants of integration. This is the basic principle that we are using in the present paper. As constants of integration, we take the classical orbit elements a, e, M_0, and ω. Thus, we will need the explicit partials of the Kepler problem solution (which is the motion of the target relative to the Earth) with respect to a, e, M_0, and ω. We give in Appendix C some additional details on how to obtain these partial derivatives. It is seen that the partials with respect to a and e are a bit complicated, although the partials in M_0 and ω are extremely simple [as seen in Eqs. (13) and (14)].

However, there is an additional complication: Our differential equations for the relative motion are in a rotating system, actually with a variable angular velocity $\dot{v} = \dot{\theta}$, such that the x axis for the relative motion frame is along the radius vector r of the target motion (Fig. 1). Therefore, we will represent the Kepler problem in polar coordinates r and θ, and we will use the 16 partial derivatives of $r, \theta, \dot{r}$, and $\dot{\theta}$ with respect to a, e, M_0, and ω. We will consider the de Vries equations (6) and (7) as a fourth-order system in the variables x, y, $\dot{x}$, and $\dot{y}$, and we will represent the solution in the form

$$x = \frac{\partial r}{\partial \alpha}, \qquad y = r\frac{\partial \theta}{\partial \alpha} \tag{B1}$$

$$\dot{x} = \frac{\partial \dot{r}}{\partial \alpha}, \qquad \dot{y} = \dot{r}\frac{\partial \theta}{\partial \alpha} + r\frac{\partial \dot{\theta}}{\partial \alpha} \tag{B2}$$

$$\ddot{x} = \frac{\partial \ddot{r}}{\partial \alpha}, \qquad \ddot{y} = \ddot{r}\frac{\partial \theta}{\partial \alpha} + 2\dot{r}\frac{\partial \dot{\theta}}{\partial \alpha} + r\frac{\partial \ddot{\theta}}{\partial \alpha} \tag{B3}$$

In these six expressions, the variable α stands for any set of four independent orbit elements. In the present work, we have used the four classical elements, a, e, M_0, and ω. The present reasoning and the canceling of terms are totally independent of which set is used. It could also be the set of equinoctial orbit elements, for instance. Note, however, that we assume that our functions are sufficiently continuous to be able to interchange time derivatives and partial derivatives, and we leave α undefined.

Substitution of these six expressions in Eqs. (6) and (7) leads to complete cancellation of all of the terms, showing that we indeed have the correct solution. Actually, note one small pitfall: that Eqs. (B3) contain the second derivatives of r and θ. It is simpler to first eliminate these second derivatives with the use of the equations of motion:

$$\ddot{\theta}r + 2\dot{r}\dot{\theta} = 0, \qquad \ddot{r} = r\dot{\theta}^2 - \mu/r^2 \tag{B4}$$

This leads us to the two new expressions for $\ddot{x}$ and $\ddot{y}$ that should be substituted in the de Vries[16] equations:

$$\ddot{x} = \dot{\theta}^2\frac{\partial r}{\partial \alpha} + 2r\dot{\theta}\frac{\partial \dot{\theta}}{\partial \alpha} + 2\frac{\mu}{r^3}\frac{\partial r}{\partial \alpha} \tag{B5}$$

$$\ddot{y} = \left(r\theta^2 - \frac{\mu}{r^2}\right)\frac{\partial \theta}{\partial \alpha} + 2\frac{\dot{r}\dot{\theta}}{r}\frac{\partial r}{\partial \alpha} - 2\dot{\theta}\frac{\partial \dot{r}}{\partial \alpha} \tag{B6}$$

Appendix C: Partial Derivatives of the Two-Body Problem

We have already said that we basically need the partial derivatives of the four two-body functions, $r, \theta, \dot{r}$, and $\dot{\theta}$, with respect to the four orbital elements, a, e, M_0, and ω. To get these partials, note that the time t and the four elements a, e, M_0, and ω are five independent quantities and that everything else depends on them, such as the three anomalies M, E, and v. Therefore, we must be careful with the exact dependence of these functions on the orbit elements. The orbit element ω is the easiest to dispose of because it enters only through the equation $\theta = \omega + v$. Therefore, the partial of θ with respect to ω is $+1$, whereas the three others, $r, \dot{r}$, and $\dot{\theta}$, vanish. We need this only to consider the elements a, e, and M_0.

The most straightforward approach is to handle the mean anomaly $M = M_0 + n(t - t_0)$ first and next the eccentric anomaly E that is connected to M thorough Kepler's equation, ending with the true anomaly and r, which are simple functions of the eccentric anomaly.

Finally, differentiation of the partials of r and θ with respect to time gives us the partials of $\dot{r}$ and $\dot{\theta}$.

To summarize, the partials of the eccentric anomaly are

$$\frac{\partial E}{\partial a} = \frac{-3n}{2r}(t - t_0), \qquad \frac{\partial E}{\partial e} = \frac{a \sin E}{r} = \frac{\sin v}{\sqrt{1 - e^2}}$$

$$\frac{\partial E}{\partial M_0} = \frac{a}{r} \tag{C1}$$

The quantities r and θ have the following six partial derivatives:

$$\frac{\partial r}{\partial a} = \frac{r}{a} - \frac{3n(t - t_0)e \sin v}{2\sqrt{1 - e^2}}$$

$$\frac{\partial r}{\partial e} = -a \cos v, \qquad \frac{\partial r}{\partial M_0} = \frac{\dot{r}}{n} \tag{C2}$$

$$\frac{\partial \theta}{\partial a} = \frac{-3n(t - t_0)\sqrt{1 - e^2}}{2a}\left(\frac{a}{r}\right)^2$$

$$\frac{\partial \theta}{\partial e} = \frac{\sin v}{1 - e^2}(2 + e \cos v), \qquad \frac{\partial \theta}{\partial M_0} = \frac{\dot{\theta}}{n} \tag{C3}$$

Note that the three partials of θ are the same as those of the true anomaly v because $\theta = \omega + v$.

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